Sample Solutions – Exercise 1

(a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{X_i^{\theta+1}} \;=\; \theta^n \exp\left(-(\theta+1)\sum_{i=1}^{n} \log X_i\right),$$

for observations satisfying $X_i \geq 1$.

We can factorize the likelihood as

$$L(\theta) = g_\theta(T)\, h(X_1, \ldots, X_n),$$

where

$$g_\theta(T) = \theta^n e^{-\theta T}, \qquad T = \sum_{i=1}^{n} \log X_i,$$

and

$$h(X_1, \ldots, X_n) = \prod_{i=1}^{n} \frac{1}{X_i} \mathbf{1}_{\{X_i \geq 1\}}.$$

By the factorization theorem,

$$\boxed{T = \sum_{i=1}^{n} \log X_i \quad \text{is sufficient for } \theta.}$$

(b) The log-likelihood function is

$$\ell(\theta) = n \log \theta - (\theta+1) \sum_{i=1}^{n} \log X_i.$$

Its first derivative is

$$\ell'(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \log X_i.$$

Setting this equal to zero gives

$$\boxed{\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} \log X_i}.}$$

The second derivative is

$$\ell''(\theta) = -\frac{n}{\theta^2} < 0.$$

Hence, the stationary point is indeed a maximum.

(c) For one observation, the log-likelihood is

$$\ell(\theta) = \log \theta - (\theta+1) \log X.$$

Its second derivative is

$$\ell''(\theta) = -\frac{1}{\theta^2}.$$

The expected Fisher information is therefore

$$I(\theta) = -E_\theta[\ell''(\theta)].$$

Consequently,

$$\boxed{I(\theta) = \frac{1}{\theta^2}.}$$

(d) By the asymptotic normality of the ML estimator,

$$\sqrt{n}(\hat{\theta}_{ML} - \theta) \xrightarrow{d} \mathcal{N}\left(0, \frac{1}{I(\theta)}\right).$$

Since $I(\theta) = 1/\theta^2$, we obtain

$$\hat{\theta}_{ML} \overset{a}{\sim} \mathcal{N}\left(\theta, \frac{\theta^2}{n}\right).$$

For the observed data,

$$\hat{\theta}_{ML} = \frac{1}{0.4} = 2.5.$$

Replacing the unknown θ in the asymptotic standard error by its ML estimate gives

$$\widehat{SE} = \frac{\hat{\theta}_{ML}}{\sqrt{n}} = \frac{2.5}{5} = 0.5.$$

An asymptotic two-sided 90% confidence interval is

$$\left[\hat{\theta}_{ML} - 1.64\widehat{SE},\ \hat{\theta}_{ML} + 1.64\widehat{SE}\right].$$

Thus,

$$\boxed{CI_{90\%} = [1.68,\ 3.32].}$$

(e) Under $H_0 : \theta = 2$, the asymptotic distribution is

$$\hat{\theta}_{ML} \overset{a}{\sim} \mathcal{N}\left(2, \frac{4}{25}\right).$$

Consequently, the test statistic

$$Z = \frac{\hat{\theta}_{ML} - 2}{2/5}$$

is asymptotically standard Gaussian under H_0.

For a two-sided test at significance level $\alpha = 0.05$, we reject H_0 if

$$|Z| > 1.96.$$

Equivalently,

$$\hat{\theta}_{ML} < 2 - 1.96 \cdot \frac{2}{5} \quad \text{or} \quad \hat{\theta}_{ML} > 2 + 1.96 \cdot \frac{2}{5}.$$

Thus, the asymptotic rejection region is

$$\boxed{\hat{\theta}_{ML} < 1.216 \quad \text{or} \quad \hat{\theta}_{ML} > 2.784.}$$

From part (d), we have

$$\hat{\theta}_{ML} = 2.5.$$

Since

$$1.216 < 2.5 < 2.784.$$

So, we do **not** reject H_0 at significance level $\alpha = 0.05$.

(f) The first derivative of $g(y)$ is $g'(y) = -\frac{1}{y^2}$.

The first-order Taylor expansion around $y_0 = 1/\theta$ gives

$$g(y) \approx g(y_0) + g'(y_0)(y - y_0).$$

We obtain

$$\boxed{\hat{\theta}_{ML} = \frac{1}{\bar{Y}} \approx \theta - \theta^2 \left(\bar{Y} - \frac{1}{\theta} \right).}$$

(g) By the central limit theorem for large n,

$$\bar{Y} \approx \mathcal{N}\left(\frac{1}{\theta}, \frac{1}{n\theta^2} \right).$$

The linear approximation is approximately normally distributed, with mean

$$\theta - \theta^2 \left(E[\bar{Y}] - \frac{1}{\theta} \right) = \theta$$

and variance

$$(-\theta^2)^2 \operatorname{Var}(\bar{Y}) = \theta^4 \frac{1}{n\theta^2} = \frac{\theta^2}{n}.$$

Consequently,

$$\boxed{\hat{\theta}_{ML} \approx \mathcal{N}\left(\theta, \frac{\theta^2}{n} \right).}$$

Sample Solutions – Exercise 2

(a) Since the two samples are independent, we have

$$\bar{X} - \bar{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}\right).$$

The observed mean difference is

$$\bar{x} - \bar{y} = 1.35 - 0.71 = 0.64.$$

The standard error is

$$SE = \sqrt{\frac{1.5}{12} + \frac{1}{8}} = \sqrt{0.25} = 0.5.$$

An exact two-sided 95% confidence interval is

$$[\bar{x} - \bar{y} - 1.96SE,\ \bar{x} - \bar{y} + 1.96SE].$$

Consequently,

$$\boxed{CI_{95\%} = [-0.34,\ 1.62].}$$

(b) Under $H_0 : \mu_D = 0$, the test statistic

$$Z = \frac{\bar{X} - \bar{Y}}{SE}$$

has a standard Gaussian distribution.

For the observed data,

$$z = \frac{0.64}{0.5} = 1.28.$$

The two-sided p-value is

$$p = P(|Z| \geq 1.28) = 2(1 - \Phi(1.28)).$$

Thus,

$$\boxed{p \approx 0.20.}$$

Sample Solutions – Exercise 3

(a) The likelihood function is

$$L(\lambda) = \prod_{i=1}^{100} \frac{\lambda^{X_i}}{X_i!} e^{-\lambda} = \frac{\lambda^S e^{-100\lambda}}{\prod_{i=1}^{100} X_i!},$$

where

$$S = \sum_{i=1}^{100} X_i.$$

The likelihood ratio statistic is defined as

$$\Lambda = \frac{L(\lambda_0)}{L(\hat{\lambda}_{ML})}, \qquad \lambda_0 = 1.$$

Substituting the likelihood function gives

$$\Lambda = \frac{e^{-100}}{\hat{\lambda}_{ML}^S e^{-100\hat{\lambda}_{ML}}}.$$

Therefore,

$$\boxed{\Lambda = \exp\Big(100(\hat{\lambda}_{ML} - 1)\Big)\, \hat{\lambda}_{ML}^{-S}.}$$

(b) Taking logarithms yields

$$-2\log\Lambda = 2\left[100(1 - \hat{\lambda}_{ML}) + S\log\hat{\lambda}_{ML}\right].$$

For the observed data,

$$S = 80, \qquad \hat{\lambda}_{ML} = 0.8.$$

Consequently,

$$\begin{aligned} -2\log\Lambda &= 2\left[20 + 80\log(0.8)\right] \\ &\approx 4.297. \end{aligned}$$

Using the asymptotic approximation, we obtain

$$p \approx P(\chi_1^2 \geq 4.297).$$

Since $Z^2 \sim \chi_1^2$ for $Z \sim \mathcal{N}(0,1)$,

$$p \approx P(|Z| \geq \sqrt{4.297}).$$

Now,

$$\sqrt{4.297} \approx 2.073 > 1.96.$$

Using Table 1, we therefore conclude that

$$\boxed{p < 0.05.}$$

Hence, we reject H_0 at significance level $\alpha = 0.05$.

Sample Solutions – Exercise 4

(a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{3} \theta^2 X_i e^{-\theta X_i} = \theta^6 \left(\prod_{i=1}^{3} X_i \right) e^{-\theta S},$$

where

$$S = X_1 + X_2 + X_3.$$

The Neyman–Pearson likelihood ratio is

$$\Lambda = \frac{L(1)}{L(2)} = \frac{e^{-S}}{2^6 e^{-2S}}.$$

Consequently,

$$\boxed{\Lambda = \frac{e^S}{64}.}$$

This likelihood ratio is strictly increasing in S.

(b) By the Neyman–Pearson lemma, the most powerful test rejects H_0 for sufficiently small values of Λ.

Since Λ is strictly increasing in S, the rejection region is of the form

$$S < c.$$

Under H_0, we have

$$S \sim \text{Gamma}(6, 1).$$

To obtain significance level $\alpha = 0.05$, we choose c such that

$$P_{\theta=1}(S < c) = 0.05.$$

Using the supplied quantile,

$$\boxed{\text{Reject } H_0 \quad \text{if} \quad S < 2.613.}$$

Equivalently, the test rejects if

$$\frac{f_0(\mathbf{x})}{f_1(\mathbf{x})} < k,$$

where the corresponding likelihood-ratio threshold is approximately

$$k = \frac{e^{2.613}}{64}.$$

(c) By construction, the Neyman–Pearson test satisfies

$$\phi^*(\mathbf{x}) = \begin{cases} 1 & \text{if } f_0(\mathbf{x}) < k f_1(\mathbf{x}), \\ 0 & \text{otherwise.} \end{cases}$$

If $f_0(\mathbf{x}) < k f_1(\mathbf{x})$, then

$$\phi^*(\mathbf{x}) - \phi(\mathbf{x}) \geq 0$$

and

$$k f_1(\mathbf{x}) - f_0(\mathbf{x}) > 0.$$

Otherwise,

$$\phi^*(\mathbf{x}) - \phi(\mathbf{x}) \leq 0$$

and

$$k f_1(\mathbf{x}) - f_0(\mathbf{x}) \leq 0.$$

Thus, in both cases,

$$\boxed{(\phi^*(\mathbf{x}) - \phi(\mathbf{x}))(k f_1(\mathbf{x}) - f_0(\mathbf{x})) \geq 0.}$$

(d) Integrating the inequality from part (c) gives

$$\begin{aligned} 0 &\leq \int (\phi^* - \phi)(k f_1 - f_0)\, d\mathbf{x} \\ &= k(E_1[\phi^*] - E_1[\phi]) \\ &\quad - (E_0[\phi^*] - E_0[\phi]). \end{aligned}$$

Consequently,

$$\begin{aligned} k(E_1[\phi^*] - E_1[\phi]) &\geq E_0[\phi^*] - E_0[\phi] \\ &= \alpha - E_0[\phi] \\ &\geq 0. \end{aligned}$$

Since $k > 0$, we conclude that

$$\boxed{E_1[\phi^*(\mathbf{X})] \geq E_1[\phi(\mathbf{X})].}$$

Therefore, the Neyman–Pearson test is most powerful.